Math 564: Advance Analysis 1
Lecture 21

We first prove a technical strengthening of the Lebesgue diff. theorem:

Technical Strengthening of Lebesgue Diff. If $f \in L^1_{loc}(\mathbb{R}^d, \lambda)$ then for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\lambda(y) = 0,$$

where $B_r(x)$ is the (open) ball at x of radius r in d_∞-metric.

Proof. We already know that for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f(y) \, d\lambda(y) = f(x).$$

For each rational $q \in \mathbb{Q}$, we also know that for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} A_r |f - q| (x) = |f(x) - q|.$$

Because $\mathbb{Q}$ is ctbl, knowing that $\forall q \in \mathbb{Q} \ \forall^{100\%} x \in \mathbb{R}^d$ implies $\forall^{100\%} x \ \forall q \in \mathbb{Q}$ (because ctbl intersection of conull is conull).

Fix $x \in \mathbb{R}^d$ for which this holds for all $q \in \mathbb{Q}$, let $q \approx_\varepsilon f(x) =: c$, so

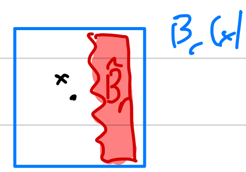
$$A_r |f - c| (x) = \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - c| \, d\lambda \leq \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f - q| \, d\lambda + \frac{1}{\lambda(B_r(x))} \int |c - q| \, d\lambda$$

$$= A_r |f - q| + |f(x) - q| \xrightarrow[r \to 0]{} |f(x) - q| + |f(x) - q| \leq 2\varepsilon. \qquad \square$$

Def. For $x \in \mathbb{R}^d$, we say that a family $\{\tilde{B}_r(x)\}_{r > 0}$ of λ-measurable sets is said to **shrink λ-nicely to x** if $\exists \, p \in (0,1)$ s.t. $\forall r > 0$

(i) $\tilde{B}_r(x) \subseteq B_r(x)$ (where this is the d_∞ ball)

(ii) $\dfrac{\lambda(\tilde{B}_r(x))}{\lambda(B_r(x))} \geq p$.



Strengthening of Leb. Diff. For each $f \in L^1_{loc}(\mathbb{R}^d)$, for λ-a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(\tilde{B}_r(x))} \cdot \int_{\tilde{B}_r(x)} |f(y) - f(x)| \, d\lambda(y) = 0,$$

for any family $\{\tilde{B}_r(x)\}_{r>0}$ that shrinks λ-nicely to x. In particular,

$$\lim_{r \to 0} \frac{1}{\lambda(\tilde{B}_r(x))} \int_{\tilde{B}_r(x)} f \, d\lambda = f(x).$$

Proof. By the technical version above, there is a conull set $X \subseteq \mathbb{R}^d$ s.t. $\forall x \in X$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\lambda(y) = 0.$$

Fix $x \in X$ and let $\{\tilde{B}_r(x)\}_{r>0}$ be any family that shrinks nicely to x. But for every $r > 0$,

$$\frac{1}{\lambda(\tilde{B}_r(x))} \int_{\tilde{B}_r(x)} |f(y) - f(x)| \, d\lambda(y) \leq \frac{1}{\rho \, \lambda(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\lambda(y) \xrightarrow[r \to 0]{} 0$$

$\square$

Lebesgue density. For a measurable set $M \subseteq \mathbb{R}^d$, the Leb. diff. theorem implies:

Cor. $d_M := \lim_{r \to 0} A_r \mathbb{1}_M = \lim_{r \to 0} \frac{\lambda(M \cap B_r(x))}{\lambda(B_r(x))} = \mathbb{1}_M$ a.e.

We call the function d_M the *Lebesgue density function* and we call the set $D_M := \{x \in \mathbb{R}^d : d_M(x) = 1\}$ the *Lebesgue density set of M*. This corollary says that $D_M =_\lambda M$, i.e. $D_M \triangle M$ is λ-null.

Strong 99% lemma. For every meas set $M \subseteq \mathbb{R}^d$ of positive Lebesgue measure, for a.e. $x \in M$, for small enough $r > 0$, M is $\geq 99\%$ of $B_r(x)$.

Also note, that if $M_0 =_\lambda M_1$ (i.e. $M_0 \triangle M_1$ is null), then $D_{M_0} = D_{M_1}$. So the map $M \mapsto D_M$ is a selector for the equiv. rel. $=_\lambda$ on the collection of measurable sets. In other words, there is a canonical representative for the $=_\lambda$-class of sets.

<u>Examples.</u> (a) If $U \subseteq \mathbb{R}^d$ is open, then $D_U = U$.

(b) If $B \subseteq \mathbb{R}^d$ is a box, then $D_B = \text{interior}(B)$.

(c) For irrationals $\mathbb{R} \setminus \mathbb{Q}$, $D_{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R}$.

(d) For any null $M \subseteq \mathbb{R}^d$, $D_M = \emptyset$.

(e) For any meas. $M \in \mathbb{R}^d$, $D_{D_M} = D_M$ (idempotent).

<u>Lebesgue density top.</u> Call a measurable set $M \subseteq \mathbb{R}^d$ Lebesgue open if $M \subseteq D_M$. Turns out that these sets form a topology, called the Lebesgue density topology. This in particular implies that arbitrary unions of Lebesgue open sets are Lebesgue measurable.
This top is not metrizable, not second ctbl or separable, but it is strong Choquet, in particular, still satisfies the Baire category theorem (nonempty open sets are nonmeager). Moreover, meager in this topology = Lebesgue null.

<u>Lebesgue density for measures.</u> Recall that applying Leb. diff. thm to Radon-Nikodym derivatives, we get:

<u>Cor.</u> Let μ be a Borel measure on $\mathbb{R}^d$ that's finite on compact sets. If $\mu \ll \lambda$, then for λ-a.e. $x \in \mathbb{R}^d$,

$$\frac{d\mu}{d\lambda}(x) = \lim_{r \to 0} \frac{\mu(\tilde{B}_r(x))}{\lambda(\tilde{B}_r(x))},$$

for any family $\{\tilde{B}_r(x)\}_{r>0}$ shrinking nicely to x.

Proof. $\mu(\tilde{B}_r(x)) = \int_{\tilde{B}_r(x)} \frac{d\mu}{d\lambda}(y)\, d\lambda(y)$. $\square$

What if $\mu \perp \lambda$? For example, let μ be the Bernoulli (p) measure on the standard Cantor set $C \cong 2^{\mathbb{N}}$. Then $\mu \perp \lambda$. For this example, $\forall x \in \mathbb{R} \setminus C$, $\lim_{r \to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} = 0$.

Turns out this is true:

Lebesgue diff. for singular measures. For each Borel measure μ on $\mathbb{R}^d$ that's finite on compact sets, if $\mu \perp \lambda$, then for λ-a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{\mu(\tilde{B}_r(x))}{\lambda(\tilde{B}_r(x))} = 0,$$

for any family $\{\tilde{B}_r(x)\}_{r>0}$ that shrinks λ-nicely to x.

Detour about locally finite measures. We have the conditions "finite on compact sets", "finite on bdd sets" show up in discussion on Borel measures on $\mathbb{R}$ and Lebesgue differentiation above. Also, in the theorem about regularity of Borel measures, we used that "openly σ-finite" condition in the hypothesis: $X = \bigcup_{n \in \mathbb{N}} U_n$, where U_n is open and finite measure.

One can prove that all these conditions are equivalent for Lebesgue measure on $\mathbb{R}^d$. Moreover, here is another equivalent condition:

Def. Let μ be a Borel measure on a top space X. Call μ **locally finite** if $\forall x \in X$ $\exists$ open neighborhood $U \ni x$ of finite μ-measure.

Then we get that for all 2^{nd} ctbl locally compact spaces all these conditions are equivalent. In particular, this is true for $\mathbb{R}^d$.

Theorem. Let μ be a Borel measure on a topological space X.

X loc. compact

X 2^{nd} ctbl

(1) μ is finite on compact sets.

(2) μ is locally finite.

(3) μ is σ-finite witnessed by open sets, i.e. $X = \bigcup_{n \in \mathbb{N}} U_n$, where each U_n is open and $\mu(U_n) < \infty$.

In English: (3) $\Rightarrow$ (2) $\Rightarrow$ (1) for all X, (1) $\Rightarrow$ (2) for locally compact X, and (2) $\Rightarrow$ (3) for 2^{nd} ctbl X.

Proof. (2) $\Rightarrow$ (1). Let $K \subseteq X$ be compact. $\forall x \in K$, $\exists$ open $U_x \ni x$ of finite μ-measure. By compactness, $\exists$ finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$ of K so, $\mu(K) < \infty$.

(3) $\Rightarrow$ (2). For each $x \in X$ $\exists\, U_n \ni x$ and $\mu(U_n) < \infty$.

(1) $\Rightarrow$ (2). Suppose X is locally compact and μ is finite on compact sets. Each $x \in X$ admits a compact neighbourhood $K \ni x$, hence $x \in \mathrm{int}(K)$ and $\mu(\mathrm{int}(K)) \leq \mu(K) < \infty$.

(2) $\Rightarrow$ (3). Suppose that admits a ctbl (open) basis $\mathcal{U}$ and μ is locally finite. Let $\mathcal{U}'$ be the subcollection of all $U \in \mathcal{U}$ that are of finite measure. Then $\mathcal{U}'$ is still a basis because for every point $x \in X$ $\exists$ open $V \ni x$ with $\mu(V) < \infty$, but V is a union of sets in $\mathcal{U}'$. Enumerate $\mathcal{U}' = (U_n)_{n \in \mathbb{N}}$ and observe that $X = \bigcup_{n \in \mathbb{N}} U_n$. $\square$

Corollary. Locally finite Borel measures on Polish spaces are strongly regular and tight.

Proof. We already know that for metric spaces being σ-finite witnessed by open sets implies strong regularity. For tightness, we only proved it for finite measures, but it extends to σ-finite measures witnessed by Polish subspaces. Our σ-finiteness is witnessed by open sets, which are Polish (in the relative topology) — this a descriptive set theory lemma. $\square$

From now on, I'll use the term locally finite.